JOURNAL OF NUMBER THEORY **18**, 135–156 (1984)

Transformation Formulae for Dirichlet Polynomials

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Communicated by H. L. Montgomery

Received January 13, 1982

Formulae of Voronoi–Atkinson type are proved for Dirichlet polynomials related to the Dirichlet series $\zeta^2(s) = \sum d(n)\, n^{-s}$ or $\varphi(s) = \sum a(n)\, n^{-s}$, where the $a(n)$ are the Fourier coefficients of a cusp form, a typical example being $a(n) = \tau(n)$, the Ramanujan function. Applications are given to a formula of Atkinson (*Acta Math.* **81** (1949), 353–376) for the mean square of $|\zeta(\frac{1}{2} + it)|$ and to the differences between consecutive zeros of $\varphi(s)$ on the critical line in the case when all the $a(n)$ are real.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The Dirichlet polynomials to be investigated in this paper are finite segments of the series,

$$\zeta^2(s) = \sum_{n=1}^{\infty} d(n)\, n^{-s}, \tag{1.1}$$

or of the series,

$$\varphi(s) = \sum_{n=1}^{\infty} a(n)\, n^{-s}, \tag{1.2}$$

where the $a(n)$ are the Fourier coefficients of a cusp form f of weight k for the full modular group. In other words, f is holomorphic in the half-plane $\operatorname{Im} \tau > 0$, $k \geqslant 12$ is an even integer, and the following equations hold for $\operatorname{Im} \tau > 0$,

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),$$

0022-314X/84 $3.00

whenever $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an integral matrix of determinant 1,

$$f(\tau) = \sum_{n=1}^{\infty} a(n)\, e^{2\pi i n \tau}.$$

According to Hecke's theory on the correspondence between modular forms and Dirichlet series, the series (1.2), which is known to be absolutely convergent in the half-plane $\sigma > \frac{1}{2}(k+1)$, can be analytically continued to an entire function satisfying the functional equation,

$$(2\pi)^{-s}\,\Gamma(s)\,\varphi(s) = (-1)^{k/2}\,(2\pi)^{s-k}\,\Gamma(k-s)\,\varphi(k-s), \tag{1.3}$$

(see, e.g., [1, Theorem 6.20]). A well-known example of $\varphi(s)$ is the Ramanujan zeta-function,

$$\sum_{n=1}^{\infty} \tau(n)\, n^{-s}, \tag{1.4}$$

where $\tau(n)$ is the Ramanujan arithmetical function; in this case $k = 12$.

We are going to study our Dirichlet polynomials on the critical line, $\sigma = \frac{1}{2}$ or $\sigma = k/2$ as the case may be, of the respective Dirichlet series (1.1) or (1.2).

Consider first Dirichlet polynomials,

$$S(M_1, M_2; t) = \sum_{M_1 \leqslant m \leqslant M_2} d(m)\, m^{-1/2-it}. \tag{1.5}$$

The main instrument in our study will be the Voronoi summation formula,

$$\sum_{a \leqslant n \leqslant b}{}' d(n) f(n) = \int_a^b (\log x + 2\gamma) f(x)\, dx + \sum_{n=1}^{\infty} d(n) \int_a^b f(x)\, \alpha(nx)\, dx, \tag{1.6}$$

where γ is Euler's constant,

$$\alpha(x) = 4K_0(4\pi x^{1/2}) - 2\pi Y_0(4\pi x^{1/2}), \tag{1.7}$$

in the standard notation of the Bessel functions (see [15]), and $\sum'$ means that if a or b is a natural number, then the corresponding term is to be halved. We need the validity of (1.6) when $0 < a < b < \infty$ and $f \in C^{(2)}[a, b]$ (see [6]).

A direct application of (1.6) to the sum (1.5) gives,

$$S(M_1, M_2; t) = \chi^2\left(\frac{1}{2} + it\right) S\left(\frac{t^2}{4\pi^2 M_2}, \frac{t^2}{4\pi^2 M_1}; -t\right) + O(\log t), \tag{1.8}$$

for $t \geqslant 2$, $1 \leqslant M_1 < M_2 \leqslant (t/2\pi)^2$, where the function $\chi(s)$ is as in the functional equation $\zeta(s) = \chi(s)\,\zeta(1-s)$, i.e.,

$$\chi(s) = \pi^{s-1/2}\Gamma(\tfrac{1}{2}(1-s))/\Gamma(\tfrac{1}{2}s).$$

By Stirling's formula we have [13, p. 68],

$$\chi(\tfrac{1}{2}+it) = (2\pi/t)^{it}\, e^{i(t+\pi/4)}\{1+O(t^{-1})\}, \tag{1.9}$$

for $t \geqslant 2$. The sum on the right of (1.8) arises when the integrals in Voronoi's formula are evaluated by the saddle-point method. But (1.8) is just what follows from the approximate functional equation (see [14]),

$$\zeta^2(s) = \sum_{n\leqslant x} d(n)\, n^{-s} + \chi^2(s) \sum_{n\leqslant y} d(n)\, n^{s-1} + O(x^{1/2-\sigma}\log t), \tag{1.10}$$

valid for $0 \leqslant \sigma \leqslant 1$, $xy = (t/2\pi)^2$, $x \geqslant 1$, $y \geqslant 1$. In fact (1.10) could be proved by Voronoi's summation formula much as the approximate functional equation of $\zeta(s)$ can be proved by Poisson's summation formula.

But there are also other possibilities to apply Voronoi's summation formula to the sum $S(M_1, M_2; t)$. The sum,

$$\sum_{M_1\leqslant m\leqslant M_2} d(m)\, m^{-(1/2)-it} e^{2\pi irm}, \tag{1.11}$$

where r is an integer is, of course, the sum $S(M_1, M_2; t)$, but the result of an application of (1.6) to this sum is of a different shape, depending on r. The most interesting case appears to be $M_1 < t/2\pi r < M_2$, especially when $r = 1$. The result is formulated in Theorem 1. For convenience we shall use the notation $A \asymp B$ to mean that $B \ll |A| \ll B$.

THEOREM 1. *Let $t \geqslant 2$, $L = \log t$, r a positive integer, and*

$$M_j = t/2\pi r + (-1)^j m_j, \qquad j = 1, 2.$$

Suppose that $r \leqslant t^{1/2-\delta}$, $m_1 \asymp m_2$, and that

$$t^{\delta}\max(t^{1/2}r^{-1}, r) \leqslant m_j \leqslant t/4\pi r, \tag{1.12}$$

where δ is a fixed positive number. Define

$$n_j = m_j^2 r^3 (t/2\pi + (-1)^j m_j r)^{-1}, \tag{1.13}$$

$$f(t; r, n) = 2t \operatorname{ar\,sinh}((\pi n/2rt)^{1/2}) + r^{-1}(\pi^2 n^2 + 2\pi nrt)^{1/2} + \pi/4. \tag{1.14}$$

Then,

$$
\begin{aligned}
&S(M_1, M_2; t)\\
&\quad = \Big\{(\log(t/2\pi r) + 2\gamma)\, r^{-1/2}\\
&\qquad + 2^{-1/2} \sum_{j=1}^{2} \sum_{n<n_j} d(n)\, e^{-\pi i n/r} n^{-1/2} \left(\frac{1}{4} + \frac{rt}{2\pi n}\right)^{-1/4} e^{i(-1)^{j-1} f(t;r,n)}\Big\}\\
&\qquad \times e^{i(-t\log(t/2\pi r)+t+\pi/4)} + O(r^{-3/2} m_1^{-1} t^{1/2} L)\\
&\qquad + O(r m_1^{1/2} t^{-1/2} L^2) + O(r^{-1/4} m_1^{-1/4} L). \qquad (1.15)
\end{aligned}
$$

Remark. Note that if $M_1 M_2 = (t/2\pi r)^2$, then by (1.13) $n_1 = n_2$, whence the second term in the curly braces in (1.15) can be written as,

$$2^{1/2} \sum_{n<n_1} d(n)\, e^{-\pi i n/r} n^{-1/2} \left(\frac{1}{4} + \frac{rt}{2\pi n}\right)^{-1/4} \cos(f(t; r, n)).$$

If also $r = 1$, then recalling (1.9) we have,

$$
\begin{aligned}
&S(M_1, M_2; t)\\
&\quad = \chi\left(\frac{1}{2} + it\right) \Big\{\log(t/2\pi) + 2\gamma\\
&\qquad + 2^{1/2} \sum_{n<n_1} (-1)^n d(n)\, n^{-1/2} \left(\frac{1}{4} + \frac{t}{2\pi n}\right)^{-1/4} \cos(f(t; 1, n))\Big\}\\
&\qquad + O(t^{1/2} m_1^{-1} L) + O(t^{-1/2} m_1^{1/2} L^2); \qquad (1.16)
\end{aligned}
$$

the error terms $O(m_1^{-1/4} L)$ and $O(t^{-2} m_1^{3/2} L)$ could be absorbed in the above error terms.

Using (1.16) and (1.10) it is now easy to prove a formula for $|\zeta(\frac{1}{2} + it)|^2$.

THEOREM 2. *Let* $t \geqslant 12\pi$, $t^\delta \ll N \leqslant t/12\pi$,

$$N' = N'(t, N) = t/2\pi + N/2 - (N^2/4 + Nt/2\pi)^{1/2}. \qquad (1.17)$$

Then,

$$
\begin{aligned}
\left|\zeta\left(\frac{1}{2} + it\right)\right|^2 &= 2^{1/2} \sum_{n\leqslant N} (-1)^n d(n)\, n^{-1/2} \left(\frac{1}{4} + \frac{t}{2\pi n}\right)^{-1/4} \cos(f(t; 1, n))\\
&\quad + 2 \sum_{n\leqslant N'} d(n)\, n^{-1/2} \cos(t\log(t/2\pi n) - t - \pi/4)\\
&\quad + O(N^{1/4} t^{-1/4} L^2 + L). \qquad (1.18)
\end{aligned}
$$

At this stage it is interesting to recall a formula of Atkinson [2]: if $N \asymp T$, then,

$$\begin{aligned}\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt \\ &= T\log(T/2\pi) + (2\gamma-1)\,T \\ &\quad + 2^{-1/2} \sum_{n\leqslant N} (-1)^n\, d(n)\, n^{-1/2}\{\operatorname{ar\,sinh}((\pi n/2T)^{1/2})\}^{-1} \\ &\quad \times (\tfrac{1}{4}+T/2\pi n)^{-1/4} \sin(f(T;1,n)) + 2 \sum_{n\leqslant N'(T,N)} d(n)\, n^{-1/2} \\ &\quad \times (\log(T/2\pi n))^{-1} \sin(T\log(T/2\pi n) - T - \pi/4) + O(\log^2 T).\end{aligned} \tag{1.19}$$

It turns out that (1.18) is a kind of a differentiated version of (1.19), with a much weaker restriction for N, however. In some applications, like the mean value estimate for $|\zeta(\frac{1}{2}+it)|^{12}$ of Heat-Brown [9], Theorem 2 can be used as a substitute for Atkinson's formula. This is of advantage because the proof of (1.18) is less complicated than that of (1.19).

Starting from (1.19) and using the argument of the proof of Theorem 1, we may relax the condition $N \asymp T$ in Atkinson's result.

THEOREM 3. *The formula* (1.19) *remains valid for* $T^\delta \ll N \ll T^2$ *with* $N' = N'(T,N)$ *as given by* (1.17) *if the error term* $O(\log^2 T)$ *in* (1.19) *is replaced by*

$$O\{(1 + T^{1/2}N^{-1} + (T/N)^{1/4})\log^2 T\}. \tag{1.20}$$

Next we turn to Dirichlet polynomials,

$$S_\varphi(M_1, M_2; t) = \sum_{M_1\leqslant m\leqslant M_2} a(m)\, m^{-k/2-it}. \tag{1.21}$$

We argue as above, using in place of (1.6) the analogous formula [3, 4],

$$\begin{aligned}\sum_{a\leqslant n\leqslant b}{}' a(n) f(n) &= 2\pi(-1)^{k/2} \sum_{n=1}^{\infty} a(n)\, n^{-(k-1)/2} \\ &\quad \times \int_a^b x^{(k-1)/2} J_{k-1}(4\pi\sqrt{nx}) f(x)\,dx,\end{aligned} \tag{1.22}$$

valid for $0 < a < b < \infty, f \in C^{(1)}[a,b]$.

By a deep result of Deligne [5], the Ramanujan–Petersson conjecture holds for the coefficients $a(n)$, i.e.,

$$a(n) \ll n^{(k-1)/2+\varepsilon}. \tag{1.23}$$

This plays for $S_\varphi(M_1, M_2; t)$ the same role as the estimate $d(n) \ll n^\varepsilon$ for $S(M_1, M_2; t)$. In applications it is sometimes possible to apply the mean value estimate of Rankin [11],

$$\sum_{n \leqslant x} |a(n)|^2 = Ax^k + O(x^{k-2/5}), \tag{1.24}$$

in place of (1.23).

We now formulate an analogue of Theorem 1 for the sum S_φ.

THEOREM 4. *Let t, r, M_j, m_j, and n_j satisfy the assumptions of Theorem 1. Then,*

$$\begin{aligned} S_\varphi(M_1, M_2; t) = 2^{-1/2} \Bigg\{ \sum_{j=1}^{2} \sum_{n < n_j} a(n)\, e^{-\pi i n/r} n^{k/2-1} \\ \times \left(\frac{1}{4} + \frac{rt}{2\pi n}\right)^{-1/4} e^{i(-1)^{j-1} f(t;r,n)} \Bigg\} \\ \times e^{i(-t\log(t/2\pi r) + t + \pi/4)} + O(rm_1^{1/2} t^{-1/2+\varepsilon}) + O(r^{-1/4} m_1^{-1/4} t^\varepsilon). \end{aligned}$$

Finally we give an application of Theorem 4 to the zeros of $\varphi(s)$ on the critical line.

THEOREM 5. *Suppose that $a(n)$ is real for all n. Then for any $\varepsilon > 0$ there exists a number $T_0 = T_0(\varepsilon)$ such that for all $T \geqslant T_0$ the function $\varphi(s)$ has a zero $k/2 + i\gamma$ with $|T - \gamma| \leqslant T^{(1/3)+\varepsilon}$.*

In particular, this theorem holds for the Ramanujan zeta-function (1.4). The method of the proof could be modified to give a similar result on the zeros of $\zeta^2(s)$, hence, also on the zeros of $\zeta(s)$, on the critical line $\sigma = \frac{1}{2}$. However, in this case much more is known, the best result being due to Karazuba [10], who proved recently that for any large T there exists a zero $\frac{1}{2} + i\gamma$ of $\zeta(s)$ such that $|T - \gamma| \leqslant T^{(5/32)+\varepsilon}$. This result is interesting because $t^{(5/32)+\varepsilon}$ is smaller than the best known estimate of $|\zeta(\frac{1}{2} + it)|$. Similarly, the bound $t^{(1/3)+\varepsilon}$ is significantly smaller than the best known estimate,

$$|\varphi(k/2 + it)| \ll t^{5/12} L^{19/12},$$

due to Good [8].

In the proof of Theorem 4 we are going to appeal to Deligne's result (1.23), and our proof of Theorem 5 will then depend on (1.23), too. However, as will be sketched after the proof of Theorem 5, by a suitable averaging device it is possible to manage with Rankin's formula (1.24).

2. Lemmas on Trigonometric Integrals

We first state a simplified version of a saddle-point lemma of Atkinson [2, Lemma 1].

LEMMA 1. *Let $f(z)$, $g(z)$ be any two functions of the complex variable z, and $[a, b]$ a real interval, such that,*

(i) *f is real and $f''(x) > 0$ in the interval $[a, b]$,*

(ii) *there exists a positive number μ such that f and g are holomorphic in the region,*

$$D = D(\mu) = \{z \mid |z - x| < \mu \text{ for some } x \in [a, b]\}, \tag{2.1}$$

(iii) *there are positive numbers F and G such that for $z \in D$,*

$$|g(z)| \ll G, \qquad |f'(z)| \ll F\mu^{-1}, \qquad |f''(z)|^{-1} \ll \mu^2 F^{-1}.$$

Let α be any real number. If the (monotonically increasing) function $f'(x) + \alpha$ has a zero in the interval (a, b), denote it by x_0. Then, if x_0 exists, we have,

$$\begin{aligned}\int_a^b g(x)\, e^{2\pi i(f(x)+\alpha x)}\, dx = {} & g(x_0) f''(x_0)^{-1/2}\, e^{2\pi i(f(x_0)+\alpha x_0)+\pi i/4} \\ & + O(G\mu F^{-3/2}) + O(G(b-a)\, e^{-A(|\alpha|\mu + F)}) \qquad (2.2) \\ & + O\{G(|f'(a)+\alpha| + f''(a)^{1/2})^{-1}\} \\ & + O\{G(|f'(b)+\alpha| + f''(b)^{1/2})^{-1}\},\end{aligned}$$

where A is a positive constant, depending on the constants implied by (iii). *If x_0 does not exist, then the two first terms on the right of* (2.2) *are to be omitted.*

We shall also need an estimate for a trigonometric integral with no saddle point, which is averaged with respect to the lower and upper limit of integration.

LEMMA 2. *Let $f(z)$ and $g(z)$ be two functions of the complex variable z such that,*

(i) *f is real in the real interval $[a, b]$,*

(ii) *there exists a positive number μ such that f and g are holomorphic in the region $D(\mu)$ (defined in* (2.1)),

(iii) *$|g(z)| \ll G$, $|f'(z)| \asymp M$ for $z \in D(\mu)$.*

Let $0 < U < \frac{1}{2}(b-a)$. *Then*

$$U^{-1}\int_0^U \left(\int_{a+u}^{b-u} g(x)\, e^{2\pi i f(x)}\, dx\right) du \ll Ge^{-A\mu M}(b-a+\mu) + GM^{-2}U^{-1}. \quad (2.3)$$

Proof. By (iii) and continuity, the derivative $f'(x)$ is of the same sign, say positive, throughout the interval $[a, b]$. Denote by $C(u)$ the contour consisting of three line segments with vertices $a+u$, $a+u+i\alpha\mu$, $b-u+i\alpha\mu$ and $b-u$, where $\alpha \in (0, \frac{1}{2}]$ is a number which will be specified in a moment. By (iii) and Cauchy's integral formula $|f''(z)| \ll M\mu^{-1}$ for $z \in D(\mu/2)$. Then, for all $z = x + yi \in C(u)$,

$$|f(z) - (f(x) + f'(x)\, yi)| \ll My^2\mu^{-1}.$$

Hence,

$$\operatorname{Im} f(x+yi) \gg My, \qquad x + yi \in C(u), \quad (2.4)$$

if α is chosen sufficiently small.

By Cauchy's integral theorem,

$$U^{-1}\int_0^U \left(\int_{a+u}^{b-u} g(x)\, e^{2\pi i f(x)}\, dx\right) du = U^{-1}\int_0^U \left(\int_{C(u)} g(z)\, e^{2\pi i f(z)}\, dz\right) du. \quad (2.5)$$

The contribution of the integral over the horizontal side of $C(u)$ is estimated by (iii) and (2.4),

$$\left|\int_{a+u+i\alpha\mu}^{b-u+i\alpha\mu} g(z)\, e^{2\pi i f(z)}\, dz\right| \ll (b-a)\, Ge^{-AM\mu}, \quad (2.6)$$

uniformly in u. It remains to consider the integrals over the vertical sides.

Changing the order of the integrations, we have,

$$\begin{aligned} &\left| U^{-1}\int_0^U \left(\int_{a+u}^{a+u+i\alpha\mu} g(z)\, e^{2\pi i f(z)}\, dz\right) du \right| \\ &\quad = \left| U^{-1}\int_0^{\alpha\mu} \left(\int_{a+iy}^{a+U+iy} g(z)\, e^{2\pi i f(z)}\, dz\right) dy \right|. \end{aligned} \quad (2.7)$$

The inner integral on the right-hand side is estimated by applying Cauchy's integral theorem to the rectangular contour with vertices $a+iy$, $a+i\alpha\mu$, $a+U+i\alpha\mu$, and $a+U+iy$. By (iii) and (2.4) we then have,

$$\left|\int_{a+iy}^{a+U+iy} g(z)\, e^{2\pi i f(z)}\, dz\right| \ll G(M^{-1}e^{-AMy} + Ue^{-AM\mu}).$$

Hence the right-hand side of (2.7) is $\ll G(M^{-2}U^{-1}+\mu e^{-AM\mu})$. Similar estimations can be made near the point b. The assertion of the lemma now follows, in the case $f'(x)>0$, on combining the last estimation with (2.5) and (2.6). In the case $f'(x)<0$ the argument is similar, except that the contours lie in the lower half-plane.

3. Proof of Theorem 1

Instead of the sum $S=S(M_1, M_2; t)$ we are going to study the average,

$$S' = U^{-1}\int_0^U S(u)\,du, \tag{3.1}$$

where

$$S(u) = \sum_{M_1+u\leqslant m\leqslant M_2-u} d(m)\, m^{-1/2-it}.$$

The parameter U will be specified later; for the moment we suppose only that for a positive constant η,

$$t^{\eta} \ll U \leqslant \tfrac{1}{2}\min(m_1, m_2). \tag{3.2}$$

We first estimate the difference,

$$S - S' \ll M_1^{-1/2}\left(\sum_{M_1\leqslant m\leqslant M_1+U} + \sum_{M_2-U\leqslant m\leqslant M_2}\right) d(m).$$

It is a well-known result that,

$$\sum_{x\leqslant n\leqslant x+y} d(n) \ll y\log x \qquad \text{for} \quad x^{\varepsilon}\ll y\ll x; \tag{3.3}$$

for a proof see Shiu [12]. By (3.3) and (3.2) we obtain,

$$S - S' \ll r^{1/2}t^{-1/2}UL. \tag{3.4}$$

Next the sum $S(u)$ is written as in (1.11), and an application on Voronoi's summation formula (1.6) gives,

$$\begin{aligned} S(u) &= \int_{M_1+u}^{M_2-u} (\log x + 2\gamma)\, x^{-1/2-it} e^{2\pi i r x}\, dx \\ &\quad + \sum_{n=1}^{\infty} d(n) \int_{M_1+u}^{M_2-u} x^{-1/2-it} e^{2\pi i r x} \alpha(nx)\, dx + O(t^{\varepsilon} M_1^{-1/2}) \\ &= I_0(u) + \sum_{n=1}^{\infty} d(n)\, I_n(u) + O(r^{1/2} t^{-1/2+\varepsilon}). \end{aligned} \tag{3.5}$$

The proof will be completed by applying Lemma 1 to the integrals $I_n(u)$ with $0 \leqslant n \leqslant n_3$ (the number n_3 will be specified in (3.10)), and estimating the averages of the other integrals $I_n(u)$ by Lemma 2.

The Integral $I_0(u)$

We apply Lemma 1 with $\alpha = r$, $f(z) = -(2\pi)^{-1} t \log z$, $g(z) = (\log z + 2\gamma) z^{-1/2}$, $\mu \asymp M_1$, $G = M_1^{-1/2} L$, $F = t$, $x_0 = t/2\pi r$, and the result is,

$$\begin{aligned} I_0(u) &= (\log(t/2\pi r) + 2\gamma)\, r^{-1/2} e^{i(-t\log(t/2\pi r) + t + \pi/4)} \\ &\quad + O(M_1^{1/2} t^{-3/2} L) + O(M_1^{1/2} e^{-At} L) \\ &\quad + O\left\{ M_1^{-1/2} L \sum_{j=1}^{2} \left(\left| r - \frac{t}{2\pi(M_j + (-1)^{j-1} u)} \right| + t^{1/2} M_1^{-1} \right)^{-1} \right\}. \end{aligned}$$

The last error term is $\ll L r^{-1/2} \min(t^{1/2} m_1^{-1} r^{-1}, 1)$, and the others are smaller. Hence,

$$\begin{aligned} U^{-1} \int_0^U I_0(u)\, du &= (\log(t/2\pi r) + 2\gamma)\, r^{-1/2} e^{i(-t\log(t/2\pi r) + t + \pi/4)} \\ &\quad + O(r^{-3/2} m_1^{-1} t^{1/2} L). \end{aligned} \tag{3.6}$$

The Integrals $I_n(u)$

Due to the known results,

$$\begin{aligned} K_0(x) &\ll e^{-x}, \\ Y_0(x) &= (2/\pi x)^{1/2} \{\sin(x - \pi/4) - (1/8x)\cos(x - \pi/4) + O(x^{-2})\}, \end{aligned}$$

valid for $x \geqslant 1$ (see [15, pp. 199, 202]), we have by (1.7),

$$\begin{aligned} \alpha(nx) &= -2^{1/2} x^{-1/4} n^{-1/4} \{\sin(4\pi\sqrt{nx} - \pi/4) \\ &\quad - (32\pi)^{-1} (nx)^{-1/2} \cos(4\pi\sqrt{nx} - \pi/4)\} + O((nx)^{-5/4}) \\ &= -2^{1/2} x^{-1/4} n^{-1/4} \sin(4\pi\sqrt{nx} - \pi/4) + O((nx)^{-3/4}). \end{aligned} \tag{3.7}$$

By the last mentioned approximation and (3.5), for $n \geqslant 1$,

$$I_n(u) = I_n^+(u) - I_n^-(u) + O(m_1 M_1^{-5/4} n^{-3/4}), \tag{3.8}$$

where

$$I_n^{\pm}(u) = i 2^{-1/2} n^{-1/4} \int_{M_1+u}^{M_2-u} x^{-3/4-it} \exp(2\pi i r x \pm i(4\pi\sqrt{nx} - \pi/4))\, dx. \tag{3.9}$$

Denote by x_0^+ and x_0^- the saddle points of these integrals, i.e., roots of the equations,

$$r - t/2\pi x \pm (n/x)^{1/2} = 0.$$

Then

$$x_0^\pm = t/2\pi r + n/2r^2 \mp r^{-2}(n^2/4 + nrt/2\pi)^{1/2}.$$

Note that $x_0^+ < t/2\pi r < x_0^-$.

A saddle point may occur for $I_n^+(u)$ if $x_0^+ > M_1$; this happens if $n < n_1$. Also, $x_0^- < M_2$ if $n < n_2$. We are going to apply Lemma 1 to $I_n^\pm(u)$ for n not exceeding,

$$n_3 = (1+\beta)\max(n_1, n_2), \tag{3.10}$$

where β is a sufficiently small positive number.

We check first that the conditions of Lemma 1 are fulfilled for $a = M_1 + u$, $b = M_2 - u$, $\alpha = r$, $f(z) = f_n^\pm(z) = -(t/2\pi)\log z \pm 2(nz)^{1/2}$, $g(z) = z^{-3/4}$, $\mu \asymp M_1$, $G = M_1^{-3/4}$, $F = t$. Let us consider the condition,

$$|(f_n^+(z))''|^{-1} \ll \mu^2 F^{-1} \qquad \text{for} \quad z \in D(\mu);$$

the others are obvious. For this to be valid it is enough to show that,

$$|t - \pi(nz)^{1/2}| \gg t \qquad \text{for} \quad z \in D(\mu), \quad n \leqslant n_3. \tag{3.11}$$

By (1.12), (1.13), and (3.10),

$$n_3 \leqslant (1+\beta)\, rt/4\pi,$$

and also

$$|z| \leqslant 3t/4\pi r + \mu \qquad \text{for} \quad z \in D(\mu).$$

Hence,

$$\pi(n|z|)^{1/2} \leqslant \tfrac{1}{2}t \qquad \text{for} \quad z \in D(\mu), \quad n \leqslant n_3,$$

if β and μM_1^{-1} are supposed to be sufficiently small, and (3.11) follows.

To calculate the saddle-point terms, note that,

$$\{(f_n^\pm)''(x_0^\pm)\}^{-1/2} = (x_0^\pm)^{3/4} n^{-1/4} \left(\frac{1}{4} + \frac{rt}{2\pi n}\right)^{-1/4},$$

and also that,

$$t(2\pi r x_0^\pm)^{-1} = \left\{ \left(\frac{\pi n}{2rt}\right)^{1/2} + \left(1 + \frac{\pi n}{2rt}\right)^{1/2} \right\}^{\pm 2},$$

whence

$$\log(t(2\pi r x_0^{\pm})^{-1}) = \pm 2 \operatorname{ar\,sinh}\left(\left(\frac{\pi n}{2rt}\right)^{1/2}\right). \tag{3.12}$$

It is then easily seen that the saddle-point terms are those given in (1.15). The saddle-point term for $I_n^+(u)$ (resp. $I_n^-(u)$) is to be taken into account if $x_0^+ > M_1 + u$ (resp. $x_0^- < M_2 - u$). However, it is more convenient to count these terms according to the simpler conditions $x_0^+ > M_1$, $x_0^- < M_2$, with an estimation of the consequent error. Write (1.13) for a moment as $n_j = n_j(m_j)$. Then the number of n's for which an extra term is counted is at most,

$$1 + n_j(m_j) - n_j(m_j - U) \ll 1 + m_1 r^3 t^{-1} U.$$

The sum of the errors is $\ll r^{-1} m_1^{-1/2} t^{\varepsilon} + r^2 m_1^{1/2} t^{-1+\varepsilon} U$. Denoting by S'' the main term on the right of (1.15), we now have by (3.1), (3.4)–(3.6), (3.8), (3.9), and the last written calculations,

$$\begin{aligned} S &= S'' + U^{-1} \int_0^U \sum_{n \leqslant n_3} d(n)(r_n^+(u) - r_n^-(u))\, du \\ &\quad + \sum_{n > n_3} d(n)\, U^{-1} \int_0^U I_n(u)\, du + R_3 \\ &= S'' + R_1 + R_2 + R_3, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} R_3 &\ll r^{1/2} t^{-1/2} UL + r^{-3/2} m_1^{-1} t^{1/2} L + r^2 m_1^{3/2} t^{-3/2} L \\ &\quad + r^{-1} m_1^{-1/2} t^{\varepsilon} + r^2 m_1^{1/2} t^{-1+\varepsilon} U, \end{aligned} \tag{3.14}$$

and $r_n^{\pm}(u)$ is the error term in the application of Lemma 1 to $I_n^{\pm}(u)$. The termwise integration of the series in (3.5) was legitimate, because the series in Voronoi's summation formula (1.6) is known to be boundedly convergent when a and b lie in a fixed closed subinterval of the interval $(0, \infty)$.

Estimation of R_1

Write $R_1 = R_1^+ - R_1^-$, where

$$R_1^{\pm} = U^{-1} \int_0^U R_1^{\pm}(u)\, du, \tag{3.15}$$

$$R_1^{\pm}(u) = \sum_{n \leqslant n_3} d(n)\, r_n^{\pm}(u). \tag{3.16}$$

Consider R_1^+ in more detail. By (3.9) and (2.2),

$$r_n^+(u) \ll n^{-1/4}(t/r)^{-3/4}\Big\{t^{-1/2}r^{-1} + m_1 e^{-At} + \sum_{j=1}^{2} (|(f_n^+)'(M_j + (-1)^{j-1}u) + r| + t^{-1/2}r)^{-1}\Big\}.$$

Multiplying both sides by $d(n)$ and summing, we may write,

$$R_1^+(u) = \sum_{i=1}^{4} R_{1i}^+(u).$$

Then

$$R_{11}^+(u) \ll r^2 m_1^{3/2} t^{-2} L, \tag{3.17}$$

and the same estimate holds trivially for $R_{12}^+(u)$.

Consider next $R_{13}^+(u)$. There exists an integer $n_0 = n_0(u) \leqslant n_3$ such that $n_0 \asymp n_3$ and $|(f_{n_0}^+)'(M_1 + u) + r|$ is a minimum. Write $n = n_0 + v$. Observing that

$$\frac{\partial}{\partial v}\{(f_n^+)'(M_1 + u)\} \asymp M_1^{-1/2} n_0^{-1/2} \qquad \text{for} \quad |v| \leqslant \frac{1}{2} n_0,$$

we may estimate,

$$\begin{aligned}
\{|(f_n^+)'(M_1 + u) + r| + t^{-1/2}r\}^{-1} \\
&\ll r^{-1} t^{1/2} && \text{for} \quad |v| \leqslant r^2 m_1 t^{-1/2}, \\
&\ll r m_1 v^{-1} && \text{for} \quad r^2 m_1 t^{-1/2} \leqslant |v| \leqslant \tfrac{1}{2} n_0, \\
&\ll r^{-2} m_1^{-1} t && \text{for} \quad |v| > \tfrac{1}{2} n_0.
\end{aligned}$$

Note that $r^2 m_1 t^{-1/2} \geqslant t^\delta$ by (1.12). Then, also using (3.3), we obtain,

$$R_{13}^+(u) \ll r m_1^{1/2} t^{-1/2} L^2. \tag{3.18}$$

The same estimate holds for $R_{14}^+(u)$, too. Moreover, the above estimates are uniform in u. Since $m_1 \ll t r^{-1}$, the right-hand side of (3.17) is smaller than that of (3.18). Hence,

$$R_1^+(u) \ll r m_1^{1/2} t^{-1/2} L^2.$$

The same estimate can be proved for $R_1^-(u)$. Thus, by (3.15),

$$R_1 \ll r m_1^{1/2} t^{-1/2} L^2. \tag{3.19}$$

Estimation of R_2

In the definition (3.5) of $I_n(u)$ we now use the more accurate version of the approximations in (3.7). Writing this as,

$$\alpha(nx) = \alpha_1(nx) + \alpha_2(nx) + O((nx)^{-5/4}).$$

we have,

$$\begin{aligned} R_2 &= \sum_{j=1}^{2} \sum_{n>n_3} d(n)\, U^{-1} \int_0^U \int_{M_1+u}^{M_2-u} x^{-1/2-it} e^{2\pi i r x} \alpha_j(nx)\, dx\, du \\ &\quad + O(rm_1^{1/2} t^{-3/2} L) \\ &= R_{21} + R_{22} + O(rm_1^{1/2} t^{-3/2} L). \end{aligned} \tag{3.20}$$

Consider first R_{21}. We apply Lemma 2 with $a = M_1$, $b = M_2$, $f(z) = f_n^{\pm}(z) + rz$, $g(z) = z^{-3/4}$, $G = M_1^{-3/4}$, $\mu \asymp M_1$, $M = n^{1/2} M_1^{-1/2}$. The assumption $|f'(z)| \asymp M$ for $z \in D(\mu)$ holds if μM_1^{-1} is supposed to be sufficiently small. We obtain,

$$\begin{aligned} R_{21} &\ll M_1^{-3/4} \sum_{n>n_3} d(n) \{ (n^{-1/4} \exp(-A n^{1/2} M_1^{1/2})\, M_1 + n^{-5/4} M_1 U^{-1} \} \\ &\ll r^{-1} m_1^{-1/2} t^{1/2} L U^{-1}. \end{aligned}$$

This estimate holds also for R_{22}, by similar arguments. Hence by (3.20),

$$R_2 \ll r^{-1} m_1^{-1/2} t^{1/2} L U^{-1} + rm_1^{1/2} t^{-3/2} L. \tag{3.21}$$

Completion of the Proof

By (3.13), (3.14), (3.19), and (3.21) we have,

$$S = S'' + R, \tag{3.22}$$

where

$$\begin{aligned} R &\ll r^{1/2} t^{-1/2} UL + r^2 m_1^{1/2} t^{-1+\varepsilon} U + r^{-3/2} m_1^{-1} t^{1/2} L \\ &\quad + rm_1^{1/2} t^{-1/2} L^2 + r^{-1} m_1^{-1/2} t^{1/2} L U^{-1}; \end{aligned} \tag{3.23}$$

the terms $r^{-1} m_1^{-1/2} t^{\varepsilon}$, $r^2 m_1^{3/2} t^{-3/2} L$, and $rm_1^{1/2} t^{-3/2} L$ could be omitted by comparison with $rm_1^{1/2} t^{-1/2} L^2$.

The first and last terms in (3.23) are equal if

$$U = r^{-3/4} m_1^{-1/4} t^{1/2}.$$

It should be verified that U satisfies (3.2). Indeed,

$$U \gg r^{-1/2} t^{1/4} \gg t^{\delta/2},$$

and by (1.12),

$$\begin{aligned} Um_1^{-1} &= r^{-3/4} m_1^{-5/4} t^{1/2} \\ &\ll r^{-3/4} t^{1/2} \min(r^{5/4} t^{(-5/8)-(5\delta/4)}, r^{-5/4}) \ll t^{-\delta}. \end{aligned}$$

With our choice of U, (3.23) now reads,

$$\begin{aligned} R \ll r^{-1/4} m_1^{-1/4} L &+ r^{5/4} m_1^{1/4} t^{(-1/2)+\varepsilon} \\ &+ r^{-3/2} m_1^{-1} t^{1/2} L + r m_1^{1/2} t^{-1/2} L^2. \end{aligned}$$

Finally, observe that the second term can be omitted by a comparison with the last one. Then the assertion (1.15) of Theorem 1 follows from (3.22).

4. Proof of Theorem 4

The proof is much similar to that of Theorem 1, so that it is not necessary to give all the details. In analogy with (1.11), the sum S_φ is written as,

$$S_\varphi(M_1, M_2; t) = \sum_{M_1 \leqslant m \leqslant M_2} a(m)\, m^{(-k/2)-it} e^{2\pi i r m}.$$

As in the proof of Theorem 1, we introduce the parameter U, and go over to an averaged sum S'_φ. The identity (1.22) is applied to sums $S_\varphi(u)$, analogous to $S(u)$. From the theory of Bessel functions, we need the asymptotic formula [15, p. 199],

$$\begin{aligned} J_\nu(x) = (2/\pi x)^{1/2} \Big\{ &\cos(x - \pi\nu/2 - \pi/4) \\ &- \frac{4\nu^2 - 1}{8x} \sin(x - \pi\nu/2 - \pi/4) \Big\} + O(x^{-5/2}). \end{aligned}$$

Hence, if k is an even integer,

$$\begin{aligned} &2\pi(-1)^{k/2} J_{k-1}(4\pi\sqrt{nx}) \\ &\quad = -2^{1/2}(nx)^{-1/4} \Big\{ \sin(4\pi\sqrt{nx} - \pi/4) \\ &\qquad + \frac{4(k-1)^2 - 1}{8x} \cos(4\pi\sqrt{nx} - \pi/4) \Big\} + O((nx)^{-5/4}). \end{aligned}$$

Note that the leading term is the same as in the formula (3.7) for $\alpha(nx)$. The sum $S_\varphi(u)$ is written in terms of integrals which are analogous to the $I_n(u)$

with $n \geqslant 1$; there is no integral corresponding to $I_0(u)$. In place of $d(n)$ we have $a(n)\,n^{-(k-1)/2}$, which is estimated by (1.23). The proof is now completed as before.

5. Proof of Theorem 2

The function,

$$Z(t) = \chi(\tfrac{1}{2}+it)^{-1/2}\,\zeta(\tfrac{1}{2}+it),$$

is real for real t, by the functional equation of $\zeta(s)$, and hence

$$|\zeta(\tfrac{1}{2}+it)|^2 = Z^2(t) = \chi(\tfrac{1}{2}+it)^{-1}\,\zeta^2(\tfrac{1}{2}+it). \tag{5.1}$$

We substitute here $\zeta^2(\frac{1}{2}+it)$ from the approximate functional equation (1.10),

$$\zeta^2(\tfrac{1}{2}+it) = \sum_{n \leqslant N'} d(n)\,n^{(-1/2)-it} + \chi^2(\tfrac{1}{2}+it) \sum_{n \leqslant N'} d(n)\,n^{(-1/2)+it}$$
$$+ \chi^2(\tfrac{1}{2}+it) \sum_{N' \leqslant n \leqslant t^2/4\pi^2 N'} d(n)\,n^{(-1/2)+it} + O(\log t).$$

Since $t/2\pi - N' \asymp (Nt)^{1/2}$ by (1.17), the assertion (1.18) follows from (5.1), (1.16), and (1.9).

6. Proof of Theorem 3

It suffices to prove that if $T^{\delta} \ll N_1 < N_2 \ll T^2$ and $N_1 \asymp N_2$, then (with $L = \log T$),

$$2 \sum_{N'(T,N_2) \leqslant n \leqslant N'(T,N_1)} d(n)\,n^{-1/2}(\log(T/2\pi n))^{-1} \sin(T\log(T/2\pi n) - T - \pi/4)$$
$$= 2^{-1/2} \sum_{N_1 \leqslant n \leqslant N_2} (-1)^n\, d(n)\, n^{-1/2} \left\{ \operatorname{ar\,sinh}\left(\left(\frac{\pi n}{2T}\right)^{1/2}\right)\right\}^{-1}$$
$$\times \left(\frac{1}{4} + \frac{T}{2\pi n}\right)^{-1/4} \sin(f(T;1,n)) + O(T^{1/2}N_1^{-1}L^2)$$
$$+ O(L^2 \min((T/N_1)^{1/2}, (T/N_1)^{1/4})) + O(N_1^{1/2}T^{-1}(T^2/N_1)^{\varepsilon}). \tag{6.1}$$

Namely, starting from Atkinson's formula (1.19) with $N \asymp T$, it is possible to shorten one sum in (1.19) and to lengthen the other by means of (6.1), in order to obtain the formula to be proved.

Consider first the case when $N_2 \leqslant N_0$, where N_0 is fixed so that $T/4\pi \leqslant N'(T, N_0) \leqslant 3T/8\pi$. As in the proof of Theorem 1 (in the case $r = 1$), we multiply the terms on the left of (6.1) by $e^{2\pi i n}(=1)$. The integrals to be calculated in Voronoi's formula are of the same type as in the proof of Theorem 1, except that there is the harmless extra factor $(\log(T/2\pi x))^{-1}$ in the integrand. As saddle points we have again the numbers x_0^+ (for $r = 1$), and the new factor $(\log(T/2\pi x_0^+))^{-1}$ in the saddle-point terms is given by (3.12). Hence, the saddle-point terms are those on the right of (6.1).

Since

$$T/2\pi - N'(T, N_1) \asymp (TN_1)^{1/2}$$

and

$$(\log(T/2\pi x))^{-1} \ll (T/N_1)^{1/2},$$

for $N'(T, N_2) \leqslant x \leqslant N'(T, N_1)$, the correct error estimate in (6.1) is obtained if the error terms of Theorem 1 with $t = T$, $r = 1$, $m_1 \asymp (N_1 T)^{1/2}$ are multiplied by $(T/N_1)^{1/2}$. The result is $\ll T^{1/2} N_1^{-1} L + (T/N_1)^{1/4} L^2 + T^{3/8} N_1^{-5/8} L$, where the last term is in any case smaller than one of the first two terms.

In the case $T \ll N_1 \ll T^2$ it is preferable to apply Voronoi's formula to the sum on the right of (6.1), or actually to an averaged sum as in the proof of Theorem 1, with the parameter,

$$U = T^{1/2} + N_1 T^{-1}.$$

The saddle-point terms are those on the left of (6.1), as is to be expected by a general reciprocity principle [7]. The error terms, which are obtained by repeating the steps of the proof of Theorem 1, contribute together $\ll T^{1/2} N_1^{-1/2} L^2 + T^{-1} N_1^{1/2} (T^2/N_1)^{\varepsilon}$. This completes the proof of (6.1).

It should be noted that the argument of Atkinson [2] works not only for $N \asymp T$, but also for $T \ll N \ll T^2$. However, the (more interesting) case $N \ll T$ makes difficulties, so that it is not obvious how to prove Theorem 3 just by modifying the argument of [2].

7. Lemmas for the Proof of Theorem 5

Lemma 3. *Let $t \geqslant 2$ and $t^2 \ll X \ll t^A$, where A is an arbitrary positive constant. Then we have,*

$$\varphi(k/2 + it) = \sum_{n \leqslant X} a(n)\, n^{(-k/2) - it} + (\log 2)^{-1} \sum_{X < n \leqslant 2X} a(n) \times \log(2X/n)\, n^{(-k/2) - it} + O(tX^{-1/2}).$$

Proof. Let $t^2 \ll x \ll t^A$. Since the series of $\varphi(s)$ is absolutely convergent for $\sigma > (k+1)/2$, by (1.24), we have

$$\sum_{n \leqslant x} a(n)\, n^{(-k/2)-it} \log(x/n)$$

$$= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \varphi(k/2+it+w)\, w^{-2} x^w \, dw$$

$$= \frac{1}{2\pi i} \int_{a-ix}^{a+ix} \varphi(k/2+it+w)\, w^{-2} x^w \, dw + O(x^{a-1}),$$

where $a \in (\frac{1}{2}, 1)$ is a fixed number. The complex integral here is evaluated by the theorem of residues, considering the integral over the rectangular contour with vertices $a \pm ix$, $-a \pm ix$. The residue at $w = 0$ is

$$\varphi(k/2+it) \log x + \varphi'(k/2+it).$$

The estimate,

$$|\varphi(\sigma+iu)| \ll (|u|+1)^{(k/2)+a-\sigma}, \qquad -a \leqslant \sigma - k/2 \leqslant a,$$

is verified as follows: the case $\sigma = k/2 + a$ is clear ($\varphi(s)$ is bounded), the case $\sigma = k/2 - a$ follows from this by the functional equation (1.3), and otherwise the assertion holds by convexity. It is now seen that the integrals over the line segments $[-a+ix, a+ix]$, $[-a-ix, a-ix]$, and $[-a-ix, -a+ix]$ contribute together $\ll t^{2a}x^{-a} + x^{a-1}$. This is $\ll tx^{-1/2}$ if we suppose that $a \leqslant \frac{1}{2} + 1/A$.

The above calculations give the equation,

$$\varphi(k/2+it)\log x + \varphi'(k/2+it)$$

$$= \sum_{n \leqslant x} a(n)\, n^{(-k/2)-it} \log(x/n) + O(tx^{-1/2}).$$

The assertion of the lemma now follows by subtracting this equation with $x = X$ from that with $x = 2X$.

LEMMA 4. *For $T \geqslant 2$, $T^{\varepsilon} \leqslant H \leqslant T$, we have*

$$\int_{T-H}^{T+H} |\varphi(k/2+it)|\, dt \gg H/\log T,$$

provided that the function $\varphi(s)$ is not identically zero.

Proof. Let m be the least index such that $a(m) \neq 0$. Define the auxiliary function,

$$\psi(s) = \varphi(s)\, m^s \exp\{((k/2 + iT - s)\, H^{-1} \log T)^2\}.$$

Then,

$$\begin{aligned}\int_{T-H}^{T+H} |\varphi(k/2+it)|\, dt &\ll \int_{T-H}^{T+H} |\psi(k/2+it)|\, dt \\ &\geqslant \left| \int_{T-H}^{T+H} \psi(k/2+it)\, dt \right| \\ &= \left| \int_{T-H}^{T+H} \psi(k/2+1+it)\, dt \right| + O(1);\end{aligned}$$

the last step was made by Cauchy's integral theorem, noting that the function $\psi(s)$ is very small for $t = T \pm H$, $k/2 \leqslant \sigma \leqslant k/2+1$.

The series $\varphi(s)$ converges absolutely on the line $\sigma = k/2+1$, whence,

$$\begin{aligned}&\int_{T-H}^{T+H} \psi(k/2+1+it)\, dt \\ &\quad = a_m \int_{T-H}^{T+H} \exp\{(-1+i(T-t))\, H^{-1} \log T)^2\}\, dt \\ &\quad\quad + m^{(k/2)+1} \sum_{n=m+1}^{\infty} a(n)\, n^{-k/2-1} \\ &\quad\quad \times \int_{T-H}^{T+H} (m/n)^{it} \exp\{((-1+i(T-t))\, H^{-1} \log T)^2\}\, dt.\end{aligned}$$

The first term on the right is $\gg H/\log T$, and by partial integration it is easily seen that the series is $\ll 1$. This completes the proof of the lemma.

8. Proof of Theorem 5

Recall the functional equation (1.3), which we write as,

$$\varphi(s) = (-1)^{k/2}\, \Delta(s)\, \varphi(k-s),$$

where

$$\Delta(s) = (2\pi)^{2s-k}\, \Gamma(k-s)/\Gamma(s). \tag{8.1}$$

Then

$$\varphi(k/2+it)\, \Delta(k/2+it)^{-1/2} = (-1)^{k/2}\, \varphi(k/2-it)\, \Delta(k/2+it)^{1/2}.$$

Ignoring the sign $(-1)^{k/2}$, the sides of this equation are complex conjugates (the assumption $a_n \in \mathbb{R}$ is needed here). Hence the function,

$$Z_\varphi(t) = \Delta(k/2 + it)^{-1/2}\, \varphi(k/2 + it), \tag{8.2}$$

is either real or purely imaginary.

Suppose now, contrary to the assertion, that $\varphi(s)$ has no zero $\rho = k/2 + i\gamma$ such that,

$$|T - \gamma| \leqslant H = T^{(1/3) + 3\varepsilon}. \tag{8.3}$$

Let $H_0 = T^{(1/3) + 2\varepsilon}$, and consider the integral,

$$I = \int_{-H}^{H} Z_\varphi(T + u)\, e^{-(u/H_0)^2}\, du. \tag{8.4}$$

The function $i^{k/2} Z_\varphi(t)$ is real and of constant sign in the interval $[T - H, T + H]$, so that by Lemma 4,

$$\begin{aligned} |I| &= \int_{-H}^{H} |Z_\varphi(T + u)|\, e^{-(u/H_0)^2}\, du \\ &\gg \int_{-H_0}^{H_0} |Z_\varphi(T + u)|\, du \gg H_0/\log T. \end{aligned} \tag{8.5}$$

On the other hand, we may estimate the integral I by substituting the expression for $\varphi(k/2 + it)$ from Lemma 3, with $X = T^3$. Writing $K = T^{(2/3) - \varepsilon}$, we have,

$$\begin{aligned} I = &\sum_{\substack{n \leqslant T^3 \\ |n - T/2\pi| > K}} a(n)\, n^{(-k/2) - iT} \int_{-H}^{H} \Delta(k/2 + i(T + u))^{-1/2}\, n^{-iu} e^{-(u/H_0)^2}\, du \\ &+ \int_{-H}^{H} \Delta(k/2 + i(T + u))^{-1/2} \\ &\times \left(\sum_{|n - T/2\pi| \leqslant K} a(n)\, n^{(-k/2) - i(T + u)} \right) e^{-(u/H_0)^2}\, du \\ &+ (\log 2)^{-1} \sum_{T^3 < n \leqslant 2T^3} a(n) \log(2T^3/n)\, n^{(-k/2) - iT} \\ &\times \int_{-H}^{H} \Delta(k/2 + i(T + u))^{-1/2}\, n^{-iu} e^{-(u/H_0)^2}\, du + O(1) \\ = &\, I_1 + I_2 + I_3 + O(1). \end{aligned} \tag{8.6}$$

We proceed to show that I_1 and I_3 are small.

Let first $n > T/2\pi + K$, and estimate the integral,

$$\int_{-H}^{H} \Delta(k/2 + i(T+u))^{-1/2} n^{-iu} e^{-(u/H_0)^2} \, du, \tag{8.7}$$

by looking at the corresponding complex integral over the rectangular contour with vertices $\pm H$, $\pm H - iH_0$. By (8.1) and Stirling's formula, for $|u| \leqslant H$,

$$\begin{aligned} &\Delta(k/2 + i(T+u))^{-1/2} n^{-iu} \\ &\qquad = \exp\{i(T \log(T/2\pi) - T + u \log(T/2\pi n) + O(1))\}. \end{aligned}$$

On the vertical sides this is bounded, and

$$e^{-(u/H_0)^2} \ll e^{-T^{\varepsilon}}.$$

On the horizontal side in the lower half-plane the function $\exp(-(u/H_0)^2)$ is bounded, and

$$\begin{aligned} \Delta(k/2 + i(T+u))^{-1/2} n^{-iu} &\ll \exp\{-H_0 \log(2\pi n/T)\} \\ &\ll \exp(-AT^{\varepsilon}). \end{aligned}$$

For $n < T/2\pi - K$ the integral (8.7) can be estimated similarly, by integrating in the upper half-plane.

By the above estimations I_1 and I_3 are found to be very small, in any case $\ll 1$.

Finally, we have,

$$\begin{aligned} I_2 &\ll H \sup_{|T-t| \leqslant H} \left| \sum_{|n - T/2\pi| \leqslant K} a(n) \, n^{(-k/2) - it} \right| \\ &\ll H \sup_{|T-t| \leqslant H} \left| \sum_{|n - t/2\pi| \leqslant K} a(n) \, n^{(-k/2) - it} \right| + O(HT^{-1/30 + 3\varepsilon/2}). \end{aligned} \tag{8.8}$$

The error term was obtained by (1.24). To estimate the sum here, we apply Theorem 4 with $r = 1$, $M_1 = t/2\pi - K$, $M_2 = t/2\pi + K$. Then $n_1, n_2 \ll t^{(1/3) - 2\varepsilon}$ by (1.13), and by Theorem 4 the sum in (8.8) is $\ll T^{-3\varepsilon/2}$. Hence by (8.8) and (8.6),

$$|I| \ll H_0 T^{-\varepsilon/2}.$$

But this contradicts (8.5) if T is sufficiently large. Consequently there must exist a zero $\rho = k/2 + i\gamma$ of $\varphi(s)$ satisfying (8.3).

Remark. The proof of Theorem 5 as given above depends on Deligne's estimate (1.23). However, it is possible to avoid (1.23) by the following argument. In the proof of Theorem 5 one may choose the parameter K freely from the interval $[T^{(2/3)-\varepsilon}, 2T^{(2/3)-\varepsilon}]$, and average with respect to K. In the above argument, it is enough to know that the sum in (8.8) is $\ll T^{-3\varepsilon/2}$ in mean. This holds if it is known that (1.23) is true in mean when n runs over an interval $[x, x + x^{(2/3)-\varepsilon}]$. But this is indeed the case by (1.24), for $\frac{3}{5} < \frac{2}{3}$.

References

1. T. M. Apostol, Modular functions and Dirichlet series in number theory, *in* "Graduate Texts in Mathematics," No. 41, Springer-Verlag, New York/Berlin, 1976.
2. F. V. Atkinson, The mean-value of the Riemann zeta function, *Acta Math.* **81** (1949), 353–376.
3. B. C. Berndt, Identities involving the coefficients of a class of Dirichlet series. V, *Trans. Amer. Math. Soc.* **160** (1971), 139–156.
4. B. C. Berndt, Identities involving the coefficients of a class of Dirichlet series. VII, *Trans. Amer. Math. Soc.* **201** (1975), 247–261.
5. P. Deligne, La conjecture de Weil, *Inst. Hautes Études Sci. Publ. Math.* **53** (1975), 273–307.
6. A. L. Dixon and W. L. Ferrar, Lattice point summation formulae, *Quart. J. Math. Oxford Ser.* (2) **2** (1931), 31–54.
7. A. L. Dixon and W. L. Ferrar, On divisor transforms, *Quart. J. Math. Oxford Ser.* (2) **3** (1932), 43–59.
8. A. Good, Beitraege zur Theorie der Dirichletreihen, die Spitzenformen zugeordnet sind, *J. Number Theory* **13** (1981), 18–65.
9. D. R. Heath-Brown, The twelfth power moment of the Riemann zeta-function, *Quart. J. Math. Oxford Ser.* (2) **29** (1978), 443–462.
10. A. A. Karazuba, On the distance between adjacent zeros of Riemann's zeta-function on the critical line, *Trudy Mat. Inst. Steklov* **157** (1981), 49–67. [Russian]
11. R. A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions. II. The order of the Fourier coefficients of integral modular forms, *Math. Proc. Cambridge Philos. Soc.* **35** (1939), 357–372.
12. P. Shiu, A Brun–Titchmarsh theorem for multiplicative functions, *J. Reine Angew. Math.* **313** (1980), 161–170.
13. E. C. Titchmarsh, "The Theory of the Riemann Zeta Function," Oxford Univ. Press (Clarendon), London/New York, 1951.
14. E. C. Titchmarsh, The approximate functional equation for $\zeta^2(s)$, *Quart. J. Math. Oxford Ser.* **9** (1938), 109–114.
15. G. N. Watson, "A Treatise on the Theory of Bessel Functions," 2nd ed., Cambridge Univ. Press, London/New York, 1944.